

# ESTIMATES ON LATTICE POINTS IN THE CIRCLE

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## Introduction

Let  $P(t)$  be the number of lattice points in the circle  $x^2 + y^2 \leq t$ . Gauss showed  $P(t) - \pi t = O(t^{\frac{1}{2}})$ . Sierpinski decreased the exponent in Gauss's result to  $1/3$ ; this followed work of Voronoi on the divisor problem. Van der Corput reduced the exponent to below  $1/3$ . There followed other improvements, Iwaniec and Mozzochi reached  $7/22$ , then further improvements by M.H.Huxley; see the works of this author for more details on the history.

This short paper provides some further estimates of this error, the best of which is

$$P(t) = \pi t + O(t^{\frac{1507}{4875}} \ln t) .$$

By classical methods, a certain sum is transformed up to error into an exponential sum similar to one also seen classically in the circle problem. This expression is then reformulated, up to error, in a way that permits cancelation of some quadratic terms in an expansion. Some applications of iterated Weyl-Van der Corput results to the restrictions of the new expression to various ranges then provide the estimate.

For the larger error estimate

$$P(t) = \pi t + O(t^{\frac{5}{16}} \ln t) ,$$

the reader can completely omit sections 4 and 6. For the exponent  $\frac{509}{1640}$ , section 4 only up to (4.8) is needed and the rest of section 4 and also section 6 can be omitted. For the exponent  $\frac{3393}{10936}$ , all of section 4 is needed but not section 6. In all cases, the logarithmic factor can be replaced by a logarithmic factor with a fractional exponent.

In 2007 Cappell and the author posted a paper on the arXiv claiming to obtain the estimate  $O(t^{\frac{1}{4}+\epsilon})$ . Unfortunately we have not been able to produce an error free version. The present paper shares with that paper the Proposition in section 5 and there is also something there akin to what immediately follows the Proposition.

## 1. A certain sum

Let  $r$  be a positive integer. A standard result, for example [T,4.7] (compare [GK 3.5]), implies

$$\sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t-m^2})}{r} = \sum_{\nu=0}^r \frac{1}{r} \int_0^{\llbracket \sqrt{t/2} \rrbracket} e(r\sqrt{t-x^2} + \nu x) dx + O(1) \quad (1.1)$$

(Actually the error could be  $O(r^{\epsilon-1})$  for  $\epsilon > 0$ .) The critical point of the phase function of the integral on the right for a given  $\nu$  is

$$x_\nu = x_{\nu,r,t} = \frac{\nu\sqrt{t}}{\sqrt{r^2 + \nu^2}}. \quad (1.2)$$

The phase function has the second derivative  $-\frac{rt}{y^3}$ , where  $y = \sqrt{t-x^2}$ . Let

$$y_\nu = y_{\nu,r,t} = \frac{r\sqrt{t}}{\sqrt{r^2 + \nu^2}} = \sqrt{t-x_\nu^2}. \quad (1.3)$$

The van der Corput estimate (e.g. [GK 3.2][T,4.4]) gives

$$\int_0^{\llbracket \sqrt{t/2} \rrbracket} e(r\sqrt{t-x^2} - \nu x) dx \ll \frac{t^{\frac{1}{4}}}{r^{\frac{1}{2}}}, \quad 0 \leq \nu \leq r \quad (1.4)$$

We have  $\frac{\partial x_\nu}{\partial \nu} = \frac{r^2 \sqrt{t}}{(r^2 + \nu^2)^{\frac{3}{2}}} \geq \frac{\sqrt{t}}{2\sqrt{2}r}$ ,  $0 \leq \nu \leq r$ ; therefore for  $1 \leq \nu \leq r-1$

$$\frac{\tau\sqrt{t}}{2\sqrt{2}r} \leq x_\nu \leq \sqrt{\frac{t}{2}} - \frac{\tau\sqrt{t}}{2\sqrt{2}r}. \quad (1.5)$$

Assuming further that  $r \leq \frac{t^{\frac{1}{2}}}{2}$ , it then follows from [GK,3.4] (compare [T,4.6]) that for  $1 \leq \nu \leq r-1$ ,

$$\int_0^{\llbracket \sqrt{t/2} \rrbracket} e(r\sqrt{t-x^2} + \nu x) = \frac{e(-1/8)rt^{\frac{1}{4}}}{(r^2 + \nu^2)^{\frac{3}{4}}} e(ry_\nu + \nu x_\nu) + O(1). \quad (1.6)$$

Therefore

$$\sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t-m^2})}{r} = \sum_{\nu=0}^r \frac{e(-1/8)t^{\frac{1}{4}}}{(r^2 + \nu^2)^{\frac{3}{4}}} e(ry_\nu + \nu x_\nu) + O\left(\frac{t^{\frac{1}{4}}}{r^{\frac{3}{2}}} + 1\right). \quad (1.7)$$

## 2. A reformulation

For  $1 \leq \nu \leq r$ , let  $m_\nu \in Z[\frac{1}{\nu}]$  be the largest rational number with denominator  $\nu$  that is at most  $x_\nu$ . Let  $m_0 = 0$ . As above, write  $y = y(x) = \sqrt{t - x^2}$ . Let  $n_\nu = \sqrt{t - m_\nu^2}$ . Let  $\lambda_\nu = x_\nu - m_\nu$ . Then  $y' = -\frac{x}{y}$  and  $y'' = -ty^{-3}$ . So  $n_0 = y_0 = \sqrt{t}$  and for  $1 \leq \nu \leq r$ ,

$$n_\nu = y_\nu + \frac{\nu}{r} \lambda_\nu + O(t^{-\frac{1}{2}} \nu^{-2}), \quad (2.1)$$

from which it follows that

$$e(ry_\nu + \nu x_\nu) = e(ry_\nu + \nu \lambda_\nu) = e(r(y_\nu + \frac{\nu}{r} \lambda_\nu)) = e(rn_\nu) + O\left(\frac{r}{\nu^2 \sqrt{t}}\right), \quad (2.2)$$

So, assuming  $r \leq t^{\frac{1}{2}}$ .

$$\sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t - m^2})}{r} = \sum_{\nu=0}^r \frac{e(1/8)t^{\frac{1}{4}}}{(r^2 + \nu^2)^{\frac{3}{4}}} e(rn_\nu) + O\left(\frac{t^{\frac{1}{4}}}{r^{\frac{3}{2}}} + 1\right); \quad (2.3)$$

the error in (2.2) gets absorbed in that of (1.7).

For a fixed  $1 < c < \sqrt{2}$ , there exist positive constants  $c_i < C_i$  (that depend on  $c$  but not  $t$ ) so that  $c_i t^{-\frac{i-1}{2}} \leq |y^{(i)}(x)| \leq C_i t^{-\frac{i-1}{2}}$ , for  $|x| \leq c^{-1}\sqrt{t}$ ; i.e.  $y^{(i)}(x) \approx t^{-\frac{i-1}{2}}$  in this range. It is also not difficult to see that for  $0 \leq \nu \leq r-1$ ,

$$\frac{\sqrt{t}}{2\sqrt{2}r} \leq x_{\nu+1} - x_\nu \leq \frac{\sqrt{t}}{r}, \quad (2.4)$$

the extremes being lower and upper bounds of  $\frac{\partial x_\nu}{\partial \nu}$ ,  $0 \leq \nu \leq r$ . Expanding around  $x_\nu$ ,

$$n_\nu - n_{\nu-1} = -\frac{\nu}{r}(m_\nu - m_{\nu-1}) + \sum_{i=2}^{N-1} \frac{y^{(i)}(x_\nu)}{i!} \left[ (-\lambda_\nu)^i - (m_{\nu-1} - x_\nu)^i \right] + O\left(\frac{\sqrt{t}}{r^N}\right) \quad (2.5.1)$$

for  $1 \leq \nu \leq r$ , and for  $0 \leq \nu \leq r-1$

$$\begin{aligned} n_{\nu+1} - n_\nu &= -\frac{\nu}{r}(m_{\nu+1} - m_\nu) + \sum_{i=2}^{N-1} \frac{y^{(i)}(x_\nu)}{i!} \left[ (m_{\nu+1} - x_\nu)^i - (-\lambda_\nu)^i \right] \\ &\quad + O\left(\frac{\sqrt{t}}{r^N}\right). \end{aligned} \quad (2.5.2)$$

The sum of these will give an expression for the difference  $n_{\nu+1} - n_{\nu-1}$ ; i.e. up to the errors this difference will be the sum of a rational number with denominator  $r$  and

$$G(\nu) = \sum_{i=2}^{N-1} \frac{y^{(i)}(x_\nu)}{i!} \left[ (m_{\nu+1} - x_\nu)^i - (m_{\nu-1} - x_\nu)^i \right]. \quad (2.6)$$

Let

$$F(\nu) = n_1 + \sum_{\mu=1}^{\frac{\nu-1}{2}} G(2\mu), \quad \nu \text{ odd}, \quad F(\nu) = n_0 + \sum_{\mu=1}^{\frac{\nu}{2}} G(2\mu-1), \quad \nu \text{ even}. \quad (2.7)$$

( $F(0) = n_0, F(1) = n_1$ ). Then from (2.5.1) and (2.5.2)

$$e(rn_\nu) = e(rF(\nu)) + O\left(\frac{\nu\sqrt{t}}{r^{N-1}}\right). \quad (2.8)$$

Hence (1.7) becomes

$$\sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t-m^2})}{r} = \sum_{\nu=0}^r \frac{e(-1/8)t^{\frac{1}{4}}}{(r^2 + \nu^2)^{\frac{3}{4}}} e(rF(\nu)) + O\left(\frac{t^{\frac{1}{4}}}{r^{\frac{3}{2}}} + 1 + \frac{t^{\frac{3}{4}}}{r^{N-\frac{3}{2}}}\right). \quad (2.9)$$

Let

$$G_1(\nu) = \sum_{i=2}^{N-1} \frac{y^{(i)}(x_\nu)}{i!} \left[ (x_{\nu+1} - x_\nu)^i - (x_{\nu-1} - x_\nu)^i \right]. \quad (2.10)$$

Then it is not difficult to see, using (2.4) and  $0 \leq \lambda_\nu \leq \nu^{-1}$ , that

$$G(\nu) - G_1(\nu) = O((r\nu)^{-1}). \quad (2.11)$$

(More definitely, one can see that there is a constant  $C$  (depending on  $N$ ) such that

$$0 < G(\nu) \leq G_1(\nu) + \frac{C}{r\nu}.)$$

Therefore, let  $F_1(\nu)$  be defined as  $F$  is in (2.7), but using  $G_1$  in place of  $G$ . Then there is an expression

$$\sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t-m^2})}{r} = \sum_{\nu=0}^r \frac{e(1/8)t^{\frac{1}{4}}}{(r^2 + \nu^2)^{\frac{3}{4}}} e(\omega_\nu) e(rF_1(\nu)) + O\left(\frac{t^{\frac{1}{4}}}{r^{\frac{3}{2}}} + 1 + \frac{t^{\frac{3}{4}}}{r^{N-\frac{3}{2}}}\right), \quad (2.12)$$

in which the coefficients  $e(\omega_\nu)$  have total variation  $O(\ln r)$ ; that is,

$$\sum_{\nu=1}^r |e(\omega_\nu) - e(\omega_{\nu-1})| \leq \sum_{\nu=1}^r \frac{C}{\nu} \ll \ln r. \quad (2.13)$$

Finally, it is not hard to see there is a positive constants  $D_i$  such that for  $0 \leq \nu \leq r$ ,

$$\left| \frac{\partial^i x_\nu}{\partial \nu^i} \right| \leq \frac{D_i \sqrt{t}}{r^i}. \quad (2.14)$$

Therefore, for  $\delta = \pm 1$  and  $1 \leq \nu \leq r-1$ ,

$$x_{\nu+\delta} - x_\nu = \sum_{k=1}^{M-1} \frac{1}{k!} \frac{\partial^k x_\nu}{\partial \nu^k} \delta^k + O\left(\frac{\sqrt{t}}{r^M}\right). \quad (2.15)$$

It then follows that  $G_1$  can be replaced by

$$G_2(\nu) = \sum_{\substack{i=3 \\ i \text{ odd}}}^{N-1} \frac{2y^{(i)}(x_\nu)}{i!} \left[ \sum_{k=1}^{N-i} \frac{1}{k!} \frac{\partial^k x_\nu}{\partial \nu^k} \right]^i \quad (2.16)$$

without further error. That is, if  $F_2(\nu)$  is defined as in (2.7) with  $G_2$  instead of  $G$  or  $G_1$ , then (2.12) still holds, i.e.

$$\sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t-m^2})}{r} = \sum_{\nu=0}^r \frac{e(-1/8)t^{\frac{1}{4}}}{(r^2 + \nu^2)^{\frac{3}{4}}} e(\omega_\nu) e(rF_2(\nu)) + O\left(\frac{t^{\frac{1}{4}}}{r^{\frac{3}{2}}} + 1 + \frac{t^{\frac{3}{4}}}{r^{N-\frac{3}{2}}}\right). \quad (2.17)$$

### 3. An Estimate

From the previous section,  $y^{(i)}(x) \approx -t^{-\frac{i-1}{2}}$  for  $i \geq 2$ ,  $|x| \leq c^{-1}\sqrt{t}$  and  $\frac{\partial^i x_\nu}{\partial \nu^i} \ll \sqrt{t}r^{-i}$ . Therefore, it is not hard to see,

$$\frac{\partial^l G_2(\nu)}{\partial \nu^l} = \frac{1}{3} \frac{\partial^l}{\partial \nu^l} \left[ y^{(3)}(x_\nu) \left( \frac{\partial x_\nu}{\partial \nu} \right)^3 \right] + A_l, \quad (3.1)$$

with

$$A_l \ll \frac{\sqrt{t}}{r^{l+4}}. \quad (3.2)$$

More explicitly,

$$\frac{\partial^l G_2(\nu)}{\partial \nu^l} = -\frac{\partial^l}{\partial \nu^l} \left[ \frac{r\nu\sqrt{t}}{(r^2 + \nu^2)^{\frac{5}{2}}} \right] + A_l, \quad (3.3)$$

In particular

$$\frac{\partial^2 G_2(\nu)}{\partial \nu^2} = -\frac{5r\nu(4\nu^2 - 3r^2)\sqrt{t}}{(r^2 + \nu^2)^{\frac{9}{2}}} + A_2. \quad (3.4)$$

Consider the sums,  $\nu$  ranging from 1 to some  $r_1 \leq r$ ,

$$S_1 = \sum_{\nu \text{ odd}} e(rF_2(\nu)) \quad S_2 = \sum_{\nu \text{ even}} e(rF_2(\nu)). \quad (3.5)$$

We now want to apply Theorem 2.6 of [GK] to estimate these sums. To do so, let  $t^\alpha < \omega < (1 - \frac{\sqrt{3}}{2})r$  for a fixed  $\alpha < 1$ . Let  $\nu_0 = \frac{\sqrt{3}}{2}r$ . Let  $I_0 = [1, \omega]$ ,  $I_1 = [\omega, \nu_0 - \omega]$ ,  $I_2 = [\nu_0 - \omega, \nu_0 + \omega]$ ,  $I_3 = [\nu_0 + \omega, r]$ . (The upper bound on  $\omega$  guarantees that this is a increasing sequence of non-empty disjoint intervals.) On the intervals  $I_1$  and  $I_3$ , from calculus methods

$$\frac{5(\sqrt{3}-1)\omega\sqrt{t}}{4\sqrt{t}} \frac{\omega\sqrt{t}}{r^6} \leq \left| \frac{r\nu(4\nu^2 - 3r^2)}{(r^2 + \nu^2)^{\frac{9}{2}}} \right| \leq 5 \frac{\sqrt{t}}{r^5} = 5 \frac{\omega\sqrt{t}}{r^6} \cdot \frac{r}{\omega}. \quad (3.6)$$

(The actual constants not important and these are not the best.) Therefore, from (3.2) and the lower bound on  $\omega$ , on these intervals

$$\frac{\omega\sqrt{t}}{r^6} \ll \left| \frac{\partial^2 G_2(\nu)}{\partial \nu^2} \right| \ll \frac{\omega\sqrt{t}}{r^6} \cdot \frac{r}{\omega}, \quad (3.7)$$

with the implied constants depending only on  $\alpha$  and the constant in (3.2). Let,  $\nu$  still confined to the range between 1 and  $r_1$ ,

$$\tilde{S}_1 = \sum_{\substack{\nu \text{ odd} \\ \nu \in I_1 \cup I_3}} e(rF_2(\nu)) \quad \tilde{S}_2 = \sum_{\substack{\nu \text{ even} \\ \nu \in I_1 \cup I_3}} e(rF_2(\nu)). \quad (3.8)$$

Then 2.6 of [GK], with the second derivative of  $G_2$  in the role of the third derivative of the phase function, would state:

$$\tilde{S}_i \ll \frac{t^{\frac{1}{12}} r^{\frac{1}{2}}}{\omega^{\frac{1}{6}}} + \frac{r}{\omega^{\frac{1}{4}}} + \frac{r^{\frac{3}{2}}}{t^{\frac{1}{8}} \omega^{\frac{1}{4}}}. \quad (3.9)$$

This estimate would follow precisely from (3.7) and (2.6) of [GK] if  $G_2$  were the derivative of  $F_2$ . However, there is just the relationship  $G_2(\nu) = F_2(\nu+1) - F_2(\nu-1)$ . It seems clear that the proof of Theorem 2.6 of [GK] goes through almost unchanged in this context. The only significant difference would be that Theorem 2.1 in [GK] as used in the proof there needs to be replaced by the stronger Kusmin-Landau inequality, which is stated for example in section 2.6 of [GK].

To be able to apply the results of [GK] exactly as stated and proved, a suitable version of the Euler-MacLaurin formula can be used as follows. Let  $I$  be an interval with first least odd integer  $\nu_1$  in  $I_1$  or  $I_3$ , and consider a sum, for example,

$$S = \sum_{\substack{\nu \text{ odd} \\ \nu \in I}} e(rF_2(\nu)). \quad (3.10)$$

Let

$$F_3(\nu) = \frac{1}{2} \int_{\nu_1}^{\nu} G_2(u) du + \sum_{j=2}^N \bar{b}_j \left( G^{(j-1)}(\nu) - G^{(j-1)}(\nu_1) \right) \quad (3.11)$$

Here  $\bar{b}_j = \frac{2^j \bar{B}_j}{j!}$ ,  $\bar{B}_j$  the values of Bernoulli polynomials at  $\frac{1}{2}$  (actually zero for  $j$  odd). From (3.3),  $\frac{\partial^N G(\nu)}{\partial \nu^N} \ll \frac{\sqrt{t}}{r^{N+3}}$ . Therefore the Euler-MacLaurin formula with remainder implies that for  $\nu$  odd

$$F_2(\nu) = F_2(\nu_1) + F_3(\nu) + O\left(\frac{\sqrt{t}}{r^{N+2}}\right). \quad (3.12)$$

Hence

$$S = e(rF_2(\nu_1)) \sum_{\substack{\nu \text{ odd} \\ \nu \in I}} e(rF_3(\nu)) + O\left(\frac{\sqrt{t}}{r^N}\right). \quad (3.13)$$

The derivative of order  $l+1$  of  $F_3(\nu)$  is one-half the derivative of  $G_2(\nu)$  of order  $l$  plus a linear combination of higher order ones. The higher order derivatives can be absorbed in  $A_l$ ; thus (3.3) holds with the derivative of order  $l$  of  $G_2$  replaced by derivative of order  $l+1$  of  $F_3$ . In particular (3.7) also holds for the third derivative of  $F_3$  on  $I \subset I_1 \cup I_3$ . Therefore,  $S$  satisfies the conclusion of Theorem (2.6) of [GK], at least possibly up to the further error (a larger error has already appeared) of  $O(\sqrt{t}r^{-N})$ . The same type of argument works for sums over even integers.

Combining (3.9) with the trivial estimate on the intervals  $I_0$  and  $I_2$  gives

$$S_i \ll \omega + \frac{t^{\frac{1}{12}} r^{\frac{1}{2}}}{\omega^{\frac{1}{6}}} + \frac{r}{\omega^{\frac{1}{4}}} + \frac{r^{\frac{3}{2}}}{t^{\frac{1}{8}} \omega^{\frac{1}{4}}}. \quad (3.14)$$

Note that the third term is at most the second if  $r \leq t^{\frac{1}{4}}$ . Set  $\omega = t^{\frac{1}{14}} r^{\frac{3}{7}}$ . Then, as long as  $(1 - \frac{\sqrt{3}}{2})^{-\frac{7}{4}} t^{\frac{1}{8}} < r \leq t^{\frac{1}{4}}$ ,

$$S_i \ll t^{\frac{1}{14}} r^{\frac{3}{7}} + \frac{r^{\frac{25}{28}}}{t^{\frac{1}{56}}}. \quad (3.15)$$

This estimate can be applied in (2.17), using (2.13) and Abel summation. Taking a large enough  $N$ , it results that for the same range of  $r$ ,

$$\sum_{m=0}^{\lfloor \sqrt{t/2} \rfloor} \frac{e(2\pi r \sqrt{t-m^2})}{r} \ll \frac{t^{\frac{9}{28}} \ln r}{r^{\frac{15}{14}}} + \frac{t^{\frac{13}{56}} \ln r}{r^{\frac{17}{28}}} + \frac{t^{\frac{1}{4}}}{r^{\frac{3}{2}}} + 1. \quad (3.16)$$

Hence, for  $(1 - \frac{\sqrt{3}}{2})^{-\frac{7}{4}} t^{\frac{1}{8}} < R_1 < R \leq t^{\frac{1}{4}}$ ,

$$\sum_{r=R_1}^R \sum_{m=0}^{\lfloor \sqrt{t/2} \rfloor} \frac{e(2\pi r \sqrt{t-m^2})}{r} \ll \frac{t^{\frac{9}{28}} \ln R}{R_1^{\frac{1}{14}}} + t^{\frac{13}{56}} R^{\frac{11}{28}} \ln R + \frac{t^{\frac{1}{4}}}{R_1^{\frac{1}{2}}} + R. \quad (3.17)$$

. On the other hand, the trivial estimate on the sum on the right in (1.8) gives (eliminating superfluous terms)

$$\sum_{r=1}^{R_1} \sum_{m=0}^{\lfloor \sqrt{t/2} \rfloor} \frac{e(2\pi r \sqrt{t-m^2})}{r} \ll t^{\frac{1}{4}} R_1^{\frac{1}{2}} + R_1. \quad (3.18)$$

So, taking for  $R_1$  the least integer greater than  $(1 - \frac{\sqrt{3}}{2})^{-\frac{7}{4}} t^{\frac{1}{8}}$  gives

$$\sum_{r=1}^R \sum_{m=0}^{\lfloor \sqrt{t/2} \rfloor} \frac{e(2\pi r \sqrt{t-m^2})}{r} \ll t^{\frac{5}{16}} \ln t + t^{\frac{13}{56}} R^{\frac{11}{28}} \ln R, \quad (3.19)$$

valid for  $1 \leq R \leq t^{\frac{1}{4}}$ .

#### 4. Further estimates

To obtain an improvement, we apply to the sums (3.5) the next case  $q = 2$  of Theorem (2.8) of [GK]. There is explicit formula

$$\frac{\partial^3 G_2(\nu)}{\partial \nu^3} = \frac{15r\sqrt{t}(8\nu^4 - 12\nu^2 r^2 + r^4)}{(r^2 + \nu^2)^{\frac{11}{2}}} + A_3. \quad (4.1)$$

The only value in  $[0, r]$  for which this derivative vanishes is  $\eta = \frac{(3-\sqrt{7})^{\frac{1}{2}}}{2} r = (1 - \frac{\sqrt{7}}{3})^{\frac{1}{2}} \nu_0$ .

Let  $J_1 = [0, \eta - \rho]$ ,  $J_2 = [\eta - \rho, \eta + \rho]$ , and  $J_3 = (\eta + \rho, r]$ . Then, similarly (slightly harder) to (3.7),

$$\frac{\rho\sqrt{t}}{r^7} \ll \left| \frac{\partial^3 G_2(\nu)}{\partial \nu^3} \right| \ll \frac{\rho\sqrt{t}}{r^7} \cdot \frac{r}{\rho} \quad (4.2)$$

on the intervals  $J_1$  and  $J_3$ , assuming  $t^\alpha < \rho < \frac{(3-\sqrt{7})^{\frac{1}{2}}}{2} r$ . Let

$$\bar{S}_1 = \sum_{\substack{\nu \text{ odd} \\ \nu \in J_1 \cup J_3}} e(rF_2(\nu)) \quad \bar{S}_2 = \sum_{\substack{\nu \text{ even} \\ \nu \in J_1 \cup J_3}} e(rF_2(\nu)). \quad (4.3)$$

Then the case  $q = 2$  and  $Q = 4$  of Theorem (2.8) of [GK] asserts:

$$\bar{S}_i \ll \frac{t^{\frac{1}{28}} r^{\frac{5}{7}}}{\rho^{\frac{1}{14}}} + \frac{r}{\rho^{\frac{1}{8}}} + \frac{r^{\frac{21}{16}}}{\rho^{\frac{1}{8}} t^{\frac{1}{16}}}. \quad (4.4)$$

Combining this with the trivial estimate over  $J_2$  gives

$$S_i \ll \rho + \frac{t^{\frac{1}{28}} r^{\frac{5}{7}}}{\rho^{\frac{1}{14}}} + \frac{r}{\rho^{\frac{1}{8}}} + \frac{r^{\frac{21}{16}}}{\rho^{\frac{1}{8}} t^{\frac{1}{16}}}. \quad (4.5)$$

Note that the last term is no larger than the previous one for  $r \leq t^{\frac{1}{5}}$ . Set  $\rho = t^{\frac{1}{30}} r^{\frac{2}{3}}$ . Then, as long as  $\frac{8}{(3-\sqrt{7})^{\frac{3}{2}}} t^{\frac{1}{10}} < r \leq t^{\frac{1}{5}}$ ,



$$S_i \ll t^{\frac{1}{30}} r^{\frac{2}{3}} + \frac{r^{\frac{11}{12}}}{t^{\frac{1}{240}}} \quad (4.6)$$

Hence, for  $\frac{8}{(3-\sqrt{7})^{\frac{3}{2}}} t^{\frac{1}{10}} < R_2 < R_1 < t^{\frac{1}{5}}$ ,

$$\sum_{r=R_2}^{R_1} \sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t-m^2})}{r} \ll t^{\frac{17}{60}} R_1^{\frac{1}{6}} \ln R_1 + t^{\frac{59}{240}} R_1^{\frac{5}{12}} \ln R_1 + \frac{t^{\frac{1}{4}}}{R_2^{\frac{1}{2}}} + R_1. \quad (4.7)$$

This can be combined with (3.17) and with the trivial estimate for  $1 \leq r \leq R_2$ . We take  $R_1$  to be the nearest integer to the minimum of  $R$  and  $t^{\frac{127}{820}}$ ; this value is obtained by equating the second term in (4.7) with the first term in (3.17), neglecting the logarithmic terms. For  $R_2$  we take the least integer more than  $\frac{8}{(3-\sqrt{7})^{\frac{3}{2}}} t^{\frac{1}{10}}$ . After eliminating superfluous terms, the result is

$$\sum_{r=1}^R \sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t-m^2})}{r} \ll t^{\frac{509}{1640}} \ln t + t^{\frac{13}{56}} R^{\frac{11}{28}} \ln R. \quad (4.8)$$

for  $R \leq t^{\frac{1}{5}}$ .

For a further improvement, we try to use a non trivial estimate on

$$\check{S}_1 = \sum_{\substack{\nu \text{ odd} \\ \nu \in J_2}} e(rF_2(\nu)) \quad \check{S}_2 = \sum_{\substack{\nu \text{ even} \\ \nu \in J_2}} e(rF_2(\nu)). \quad (4.9)$$

Explicitly,

$$\frac{\partial^4 G_2(\nu)}{\partial \nu^4} = -\frac{105r\nu\sqrt{t}(8\nu^4 - 20\nu^2 r^2 + 5r^4)}{(r^2 + \nu^2)^{\frac{13}{2}}} + A_4. \quad (4.10)$$

The first term on the right is zero in the relevant range only at

$$\nu = \frac{(5 - \sqrt{15})^{\frac{1}{2}}}{2} r,$$

which is distinct multiple of  $r$  from  $\eta = \frac{(3-\sqrt{7})^{\frac{1}{2}}}{2} r$ . Another calculus type argument shows that, as long as  $\rho < kr$  ( $k$  say half the difference of the two above multipliers of  $r$ ),

$$\frac{\partial^4 G_2(\nu)}{\partial \nu^4} \approx \frac{\sqrt{t}}{r^7} \quad (4.11)$$

on the interval  $J_2$ . Then Theorem (2.8) of [GK] with  $q = 3$  and  $Q = 8$  asserts

$$\check{S}_i \ll \frac{\rho t^{\frac{1}{60}}}{r^{\frac{1}{5}}} + \rho^{\frac{15}{16}} + \frac{\rho^{\frac{49}{64}} r^{\frac{3}{8}}}{t^{\frac{1}{32}}}. \quad (4.12)$$

Therefore, if  $t^\alpha < \rho < kr$ , and  $r \leq t^{\frac{1}{5}}$ ,

$$S_i = \bar{S}_i + \check{S}_i \ll \frac{t^{\frac{1}{28}} r^{\frac{5}{7}}}{\rho^{\frac{1}{14}}} + \frac{r}{\rho^{\frac{1}{8}}} + \frac{\rho t^{\frac{1}{60}}}{r^{\frac{1}{5}}} + \rho^{\frac{15}{16}} + \frac{\rho^{\frac{49}{64}} r^{\frac{3}{8}}}{t^{\frac{1}{32}}}. \quad (4.13)$$

Set  $\rho = t^{\frac{2}{25}} r^{\frac{152}{375}}$ ; this equalizes the first and last terms. Then, as long as  $k_1 t^{\frac{30}{223}} < r \leq t^{\frac{1}{5}}$ , for a suitable constant  $k_1$ ,

$$S_i \ll t^{\frac{3}{100}} r^{\frac{257}{375}} + \frac{r^{\frac{356}{375}}}{t^{\frac{1}{100}}} + t^{\frac{29}{300}} r^{\frac{77}{375}} + t^{\frac{3}{40}} r^{\frac{19}{50}}. \quad (4.14)$$

Hence, for  $0 < k_1 t^{\frac{30}{223}} < R_2 < R_1 < t^{\frac{1}{5}}$ ,

$$\begin{aligned} \sum_{r=R_2}^{R_1} \sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t-m^2})}{r} &\ll t^{\frac{7}{25}} R_1^{\frac{139}{750}} \ln R_1 + t^{\frac{6}{25}} R_1^{\frac{337}{750}} \ln R_1 + \frac{t^{\frac{52}{150}} \ln R_1}{R_2^{\frac{221}{750}}} \\ &\quad + \frac{t^{\frac{13}{40}} \ln R_1}{R_2^{\frac{3}{25}}} + \frac{t^{\frac{1}{4}}}{R_2^{\frac{1}{2}}} + R_1. \end{aligned} \quad (4.15)$$

This can be combined with (3.17). Take for  $R_1$  the nearest integer to  $t^{\frac{855}{5648}}$ ; this results from equating the second term in (4.15) and the first of (3.17), neglecting logarithmic terms. With the substitution, the first term in (4.15) is slightly smaller than the second (equal to the last), and the last two terms of (3.17) and (4.15) are also superfluous, so for  $0 < k_1 t^{\frac{30}{223}} < R_2 < R < t^{\frac{1}{5}}$ ,

$$\sum_{r=R_2}^R \sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t-m^2})}{r} \ll t^{\frac{3393}{10936}} \ln t + t^{\frac{13}{56}} R^{\frac{11}{28}} \ln R + \frac{t^{\frac{52}{150}} \ln R_1}{R_2^{\frac{221}{750}}} + \frac{t^{\frac{13}{40}} \ln R_1}{R_2^{\frac{3}{25}}}. \quad (4.16)$$

Now take  $R_2$  to be the first integer more than  $k_1 t^{\frac{30}{223}}$ , apply (4.7) in the range  $t^{\frac{1}{10}} < r \leq k_1 t^{\frac{30}{223}}$  and the trivial estimate below this range. All the terms obtained, i.e. the last two in (4.16) and what we get from (4.7) and the trivial estimate, are majorized by the first term of (4.16). Therefore

$$\sum_{r=1}^R \sum_{m=0}^{\llbracket \sqrt{t/2} \rrbracket} \frac{e(2\pi r \sqrt{t-m^2})}{r} \ll t^{\frac{3393}{10936}} \ln t + t^{\frac{13}{56}} R^{\frac{11}{28}} \ln R, \quad (4.17)$$

$$1 \leq R \leq t^{\frac{1}{5}}.$$

## 5. Estimates on $P(t) - \pi t$ .

Let  $\psi(t) = t - \llbracket t \rrbracket - \frac{1}{2}$  be the usual saw-tooth function. This section first (re)states analogues of well known results for the divisor problem.

**Proposition.**

$$\sum_0^{\llbracket \sqrt{t/2} \rrbracket} \psi(\sqrt{t-m^2}) = \frac{\pi t - P(t)}{8} + O(1) . \quad (5.1)$$

Proof:

By trapezoidal approximation using integer points from 0 to the greatest integer  $\alpha$  in  $\sqrt{\frac{t}{2}}$  ,

$$\sum_0^\alpha \psi(\sqrt{t-m^2}) = A - L + \frac{\sqrt{t}}{2} + O(1) , \quad (5.2)$$

where  $A$  is the area under the circle of radius  $y = \sqrt{t-x^2}$  for  $0 \leq x \leq \alpha$  and  $L$  the number of lattice points in this region, counting the top and sides but not the bottom. Combining this with "Pick's theorem"

$$L_0 = A_0 + \frac{3}{2}\alpha + 1 \quad (5.3)$$

for the number of lattice points  $L_0$  in the triangle with vertices the origin,  $(\alpha, 0)$  , and  $(\alpha, \alpha)$  , and with area  $A_0$  , it follows that

$$\sum_1^\alpha \psi(\sqrt{t-m^2}) = \frac{\pi t}{8} - L_1 + \frac{\sqrt{t}}{2} - \frac{1}{2}\sqrt{\frac{t}{2}} + O(1) , \quad (5.4)$$

$L_1$  the number of lattice points contained in the sector of the circle of radius  $\sqrt{t}$  with angle  $\frac{\pi}{4} < \theta \leq \frac{\pi}{2}$  . However, it is obvious that the number of lattice points in the circle is

$$P(t) = 8L_1 + 4\sqrt{\frac{t}{2}} - 4\sqrt{t} , \quad (5.5)$$

and the result follows.

Let  $\|u\|$  be the distance from a real number  $u$  to the nearest integer. The Fourier expansion of  $\psi$  gives

$$\sum_0^{\sqrt{t/2}} \psi(\sqrt{t-m^2}) = -\frac{1}{\pi} \sum_0^{\sqrt{t/2}} \sum_{r=1}^\infty \frac{\sin 2\pi r \sqrt{t-m^2}}{r} , \quad (5.3)$$

Let  $J$  denote the set of lattice points  $(m, n)$  with  $0 \leq m \leq \alpha$  , ( $\alpha$  as in the previous proof) and with  $\sqrt{t-m^2} - \frac{1}{2} \leq n < \sqrt{t-m^2} + \frac{1}{2}$  . Then

$$\begin{aligned}
\sum_0^{\sqrt{t/2}} \sum_{r=R+1}^{\infty} \frac{\sin 2\pi r \sqrt{t-m^2}}{r} &\ll \sum_0^{\sqrt{t/2}} \min \left\{ \frac{\|\sqrt{t-m^2}\|^{-1}}{R}, 1 \right\} \\
&\leq \sum_{(m,n) \in J} \min \left\{ \frac{1}{R|\sqrt{m^2+n^2}-\sqrt{t}|}, 1 \right\},
\end{aligned} \tag{5.4}$$

The last inequality follows from the fact the closest point on a circle to a given point is the intersection with the circle of the line joining that point to the center of the circle. This fact also means that if  $J_k$  consists of those points in  $J$  with  $\frac{k}{2R} \leq \sqrt{m^2+n^2}-\sqrt{t} < \frac{k+1}{2R}$ , then

$$J = \bigcup_{k=-R}^{R-1} J_k. \tag{5.5}$$

Therefore

$$\sum_0^{\sqrt{t/2}} \sum_{r=R+1}^{\infty} \frac{\sin 2\pi r \sqrt{t-m^2}}{r} \ll \sum_{k=-R}^{-2} \frac{n(J_k)}{-k-1} + \sum_{k=2}^{R-1} \frac{n(J_k)}{k} + n(J_{-1}) + n(J_0), \tag{5.6}$$

$n(J_k)$  the number of elements of  $J_k$ . Further, distinct circles of radius less than  $\sqrt{s}$  that contain lattice points must differ in radius by at least  $\frac{1}{2\sqrt{s}}$ ; as is well known, each such circle has at most  $O(s^\epsilon)$  lattice points, any  $\epsilon > 0$ . Therefore  $n(J_k) \ll t^{\frac{1}{2}+\epsilon} R^{-1}$ , assuming  $R \leq t^{\frac{1}{2}}$ , and so, also using  $\ln t \ll t^\epsilon$ ,

$$\pi t - P(t) = -\frac{8}{\pi} \sum_{m=0}^{\sqrt{\frac{t}{2}}} \sum_{r=1}^R \frac{\sin 2\pi r \sqrt{t-m^2}}{r} + O\left(\frac{t^{\frac{1}{2}+\epsilon}}{R}\right). \tag{5.7}$$

Therefore, for  $R < t^{\frac{1}{5}}$ ,

$$P(t) - \pi t \ll t^\beta \ln t + t^{\frac{13}{56}} R^{\frac{11}{28}} \ln R + \frac{t^{\frac{1}{2}+\epsilon}}{R}. \tag{5.8}$$

with  $\beta = \frac{5}{16}$  from (3.19),  $\beta = \frac{509}{1640}$  from (4.8), or  $\beta = \frac{3393}{10936}$  from (4.17) Choose  $\epsilon$  small and  $R$  to be the nearest integer to  $t^{\frac{15}{78}}$ ; then all the other terms are less than the first, i.e.

$$P(t) - \pi t \ll t^\beta \ln t. \tag{5.9}$$

## 6. A further estimate

In place of the trivial estimate on the intervals  $I_0$  and  $I_2$  used to obtain (3.14), observe that for  $\omega < k_2 r$ , for a suitable constant  $k_2$ ,

$$\frac{\partial^3 G_2(\nu)}{\partial \nu^3} \approx \frac{\sqrt{t}}{r^6} \quad (6.1)$$

on these intervals. Therefore, we can apply Theorem (2.8) of [GK] with  $q = 2$  to replace (3.14) with

$$S_i \ll \frac{\omega t^{\frac{1}{28}}}{r^{\frac{5}{14}}} + \omega^{\frac{7}{8}} + \frac{\omega^{\frac{9}{16}} r^{\frac{5}{8}}}{t^{\frac{1}{16}}} + \frac{t^{\frac{1}{12}} r^{\frac{1}{2}}}{\omega^{\frac{1}{6}}} + \frac{r}{\omega^{\frac{1}{4}}} + \frac{r^{\frac{3}{2}}}{t^{\frac{1}{8}} \omega^{\frac{1}{4}}}, \quad (6.2)$$

$t^\alpha < \omega < k_3 r$ . Set  $\omega = t^{\frac{2}{25}} r^{\frac{12}{25}}$ , the result of equating the second and fourth terms. Then, as long as  $k_4 t^{\frac{2}{13}} < r \leq t^{\frac{1}{4}}$ , for a suitable  $k_4$ ,

$$S_i \ll t^{\frac{81}{700}} r^{\frac{43}{350}} + t^{\frac{7}{100}} r^{\frac{21}{50}} + \frac{r^{\frac{179}{200}}}{t^{\frac{7}{400}}} + \frac{r^{\frac{22}{25}}}{t^{\frac{1}{50}}}. \quad (6.3)$$

Note the last term is majorized by the previous one and the first by the second.

Therefore, analogous to previous sections, for  $k_4 t^{\frac{2}{13}} < R_1 < R < t^{\frac{1}{4}}$ ,

$$\sum_{r=R_1}^R \sum_{m=0}^{\ll \sqrt{t/2}} \frac{e(2\pi r \sqrt{t-m^2})}{r} \ll \frac{t^{\frac{8}{25}} \ln t}{R_1^{\frac{2}{25}}} + t^{\frac{93}{400}} R^{\frac{79}{200}} \ln t + \frac{t^{\frac{1}{4}}}{R_1^{\frac{1}{2}}} + R. \quad (6.4)$$

Now suppose  $k_4 t^{\frac{2}{13}} < R_1 < R < t^{\frac{1}{5}}$ ,  $k_2 t^{\frac{30}{223}} < R_2 < R_1$ . Then by (6.4), (4.15), (4.7) in the range  $\frac{8}{(3-\sqrt{7})^{\frac{3}{2}}} t^{\frac{1}{10}} < r \leq R_2$ , and the trivial estimate below this range,

$$\begin{aligned} \sum_{r=1}^R \sum_{m=0}^{\ll \sqrt{t/2}} \frac{e(2\pi r \sqrt{t-m^2})}{r} &\ll \frac{t^{\frac{8}{25}} \ln t}{R_1^{\frac{2}{25}}} + t^{\frac{93}{400}} R^{\frac{79}{200}} \ln t + t^{\frac{7}{25}} R_1^{\frac{139}{750}} \ln R_1 + t^{\frac{6}{25}} R_1^{\frac{337}{750}} \ln R_1 \\ &+ \frac{t^{\frac{52}{150}} \ln R_1}{R_2^{\frac{221}{750}}} + \frac{t^{\frac{13}{40}} \ln R_1}{R_2^{\frac{3}{25}}} + t^{\frac{17}{60}} R_2^{\frac{1}{6}} \ln R_2 + t^{\frac{59}{240}} R_2^{\frac{5}{12}} \ln R_2 + t^{\frac{3}{10}}. \end{aligned}$$

Let  $R_1$  be the first integer at least  $k_4 t^{\frac{2}{13}}$  and  $R_2$  the nearest integer to  $t^{\frac{95}{692}}$  (gotten by equating the fifth and seventh terms). Then the fourth term majorizes everything else except possibly the second, and we get:

$$\sum_{r=1}^R \sum_{m=0}^{\ll \sqrt{t/2}} \frac{e(2\pi r \sqrt{t-m^2})}{r} \ll t^{\frac{1815}{5876}} \ln t + t^{\frac{93}{400}} R^{\frac{79}{200}} \ln t. \quad (6.5)$$

Note that if using (6.4) with a smaller lower bound could be justified, the estimate could be improved further. In any case, as in the previous section,

$$P(t) - \pi t \ll t^{\frac{1507}{4875}} \ln t + t^{\frac{93}{400}} R^{\frac{79}{200}} \ln t + \frac{t^{\frac{1}{2}+\epsilon}}{R}. \quad (6.6)$$

Take  $R$  the nearest integer to  $t^{\frac{107}{558}}$  and  $\epsilon$  small. Then the other terms are less than the first, i.e.

$$P(t) - \pi t \ll t^{\frac{1507}{4875}} \ln t. \quad (6.7)$$

Concluding remark.

The method of exponent pairs, as presented in [GK] and elsewhere, does not apply directly to the exponential sums considered here. Rather, results like those quoted produce exponent systems as originally defined by Van der Corput. However, methods used to get various exponents pairs should be able to be extended to produce further exponent systems beyond the ones quoted here and thereby possibly also produce better estimates for the circle problem.

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